Condensed Matter Physics

Statistical physics of dislocations

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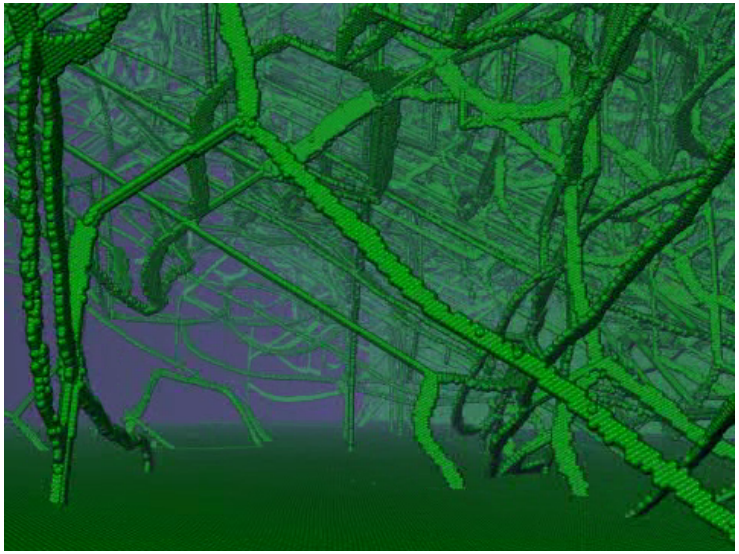
ELTE

May 3, 2019

Microscopic picture



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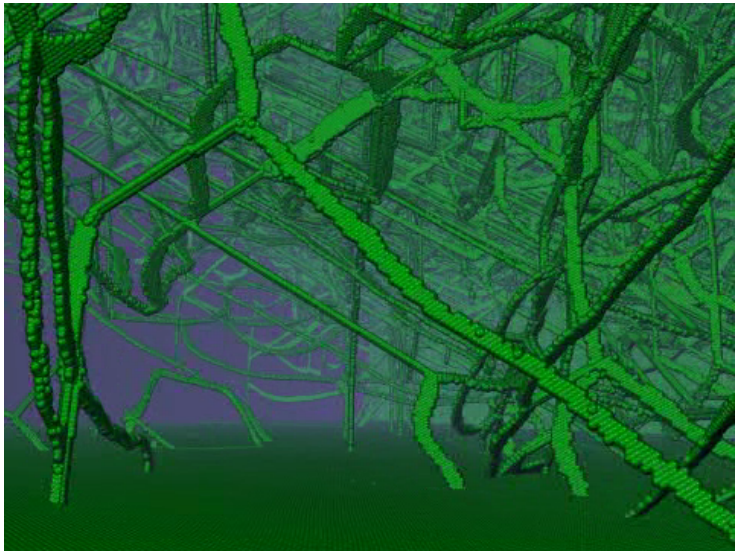

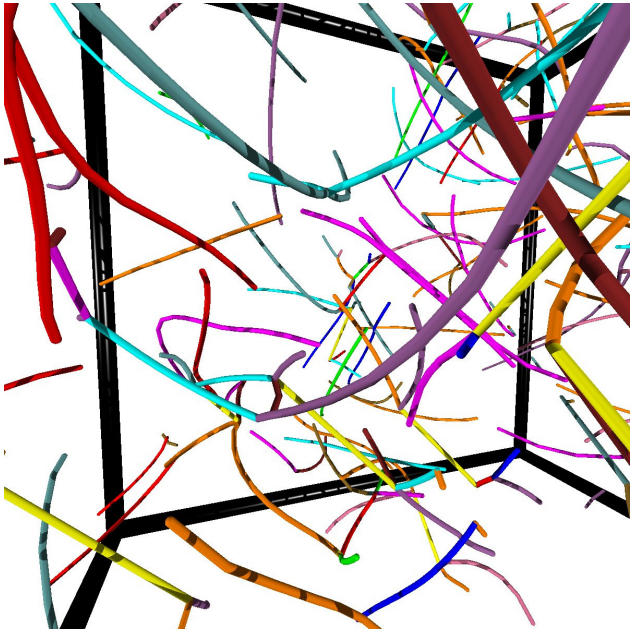

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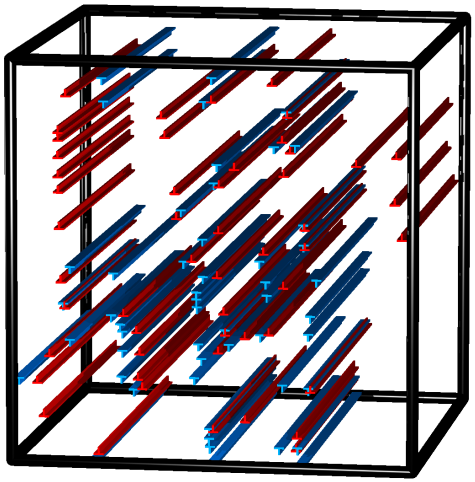

DDD single slip in 2D

Equation of motion

$$\frac{dx_i}{dt} = M_0 b \left(\sum_{j=1}^{N} s_i s_j \tau_{ind}(\vec{r}_j - \vec{r}_i) + \tau_{ext} \right)$$

DDD single slip in 2D

Equation of motion with thermal noise

$$\frac{dx_i}{dt} = M' \frac{b}{\sqrt{\rho}} \left(\sum_{j=1}^{N} s_i s_j \tau_{ind}(\vec{r}_j - \vec{r}_i) + \tau_{ext} \right) + \sqrt{2M' k_B T} \zeta_i(t)$$

$$< \zeta_i(t) \zeta_j(0) >= \delta_{ij} \delta(t)$$

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Since

$$\tau_{ind} = \partial_x \partial_y \chi_{ind} \quad \Rightarrow \quad \tau_{ind} = -\partial_x \Phi_{ind}$$

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$$\tau_{ind} = \partial_x \partial_y \chi_{ind} \quad \Rightarrow \quad \tau_{ind} = -\partial_x \Phi_{ind}$$

$$\frac{dx_i}{dt} = -M' \partial_{x_i} V(\{\vec{r}_i\}) + \sqrt{2M' k_B T} \zeta_i(t)$$

$$V(\{\vec{r}_i\}) = \frac{b}{2\sqrt{\rho}} \sum_{ij} s_i s_j \Phi_{ind}(\vec{r}_i - \vec{r}_j)$$

Fokker–Planck equation

$$\frac{dp(\{x_i\},t)}{dt} = M' \sum_i \partial_{x_i} \left[p(\{x_i\},t) \partial_{x_i} V(\{\vec{r}_i\}) + k_B T \partial_{x_i} p(\{x_i\},t) \right]$$

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Steady state solution

$$\left[\partial_{x_i} V(\{\vec{r}_i\}) + k_B T \partial_{x_i} \right] p_\infty(\{x_i\}) = 0$$

$$p_\infty(\{x_i\}) = \frac{1}{Z} e^{-\frac{V}{k_B T}}$$

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Time scales

$$t_{relax} = \frac{1}{M'\rho} \frac{\sqrt{\rho}}{b^2 \mu} = \frac{1}{M_0 \rho b^2 \mu} \quad \ll \quad t_T = \frac{1}{M'\rho} \frac{1}{k_B T}$$

Time evolution of the energy

$$\frac{dV}{dt} = \sum_i \vec{v}_i \frac{dV}{d\vec{r}_i} = -\sum_i M' v_i^2 < 0$$

Coarse graining

Set a parallel edge dislocations with $\pm\vec{b}$

$$\nabla^4\chi = \frac{2\mu}{1-\nu}\left(b\partial_y\right)\kappa_d \quad \text{with} \quad \kappa_d = \sum_i s_i\delta(\vec{r}-\vec{r}_i)$$

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Convolution with $w(\vec{r})$

$$\int w(\vec{r}-\vec{r}')\nabla'^4\chi(\vec{r}')d\vec{r}' = \frac{2\mu}{1-\nu}\int w(\vec{r}-\vec{r}')b\partial_{y'}\kappa_d(\vec{r}')d\vec{r}'$$

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Partial integrations

$$\nabla^4 \int w(\vec{r}-\vec{r}')\chi(\vec{r}')d\vec{r}' = \frac{2\mu}{1-\nu} b\partial_y \int w(\vec{r}-\vec{r}')\kappa_d(\vec{r})d\vec{r}'$$

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Coarse grained fields

$$<\chi>= \int w(\vec{r}-\vec{r}')\chi(\vec{r}')d\vec{r}' \qquad <\kappa>= \int w(\vec{r}-\vec{r}')\kappa_d(\vec{r}')d\vec{r}'$$

$$<\sigma>_{11}= -\partial_{yy}<\chi>, \qquad <\sigma>_{22}= -\partial_{xx}<\chi>, \qquad <\sigma>_{12}= \partial_{xy}<\chi>$$

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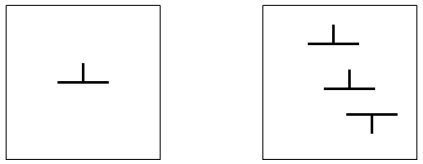

$Micro -> meso$

Equation of motion

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$$-\frac{d}{dt}\rho_{disc}(\vec{r}) = M_0 \vec{b} \frac{d}{d\vec{r}} \left\{ \int \tau_{ind}(\vec{r}')\rho_{disc}(\vec{r}')\rho_{disc}(\vec{r} - \vec{r}')d\vec{r}' + \tau_{ext}\rho_{disc}(\vec{r}) \right\},$$

$$\rho_{disc}(\vec{r}, \vec{r}_1, .., \vec{r}_N) \quad = \quad \sum_{i=1}^{N} \delta(\vec{r} - \vec{r}_i)$$

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Coarse graining

$$\rho_1(\vec{r}) =< \rho_{disc}(\vec{r}) >_{CG}, \quad \rho_2(\vec{r}, \vec{r}') =< \rho_{disc}(\vec{r}')\rho_{disc}(\vec{r} - \vec{r}') >_{CG}$$

$Micro - > meso$

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$$\frac{\partial \rho_1(\vec{r}_1, t)}{\partial t} + M_0 \vec{b} \frac{d}{d\vec{r}_1} \left[\rho_1(\vec{r}_1, t)\tau_{\rm ext} + \int \rho_2(\vec{r}_1, \vec{r}_2, t)\tau_{\rm ind}(\vec{r}_1 - \vec{r}_2)\mathrm{d}\vec{r}_2 \right] = 0$$

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$$\frac{\partial \rho_+(\vec{r}_1, t)}{\partial t} + M_0 \vec{b} \frac{d}{d\vec{r}_1} \left[\rho_+(\vec{r}_1, t)\tau_{\rm ext} + \int \left\{ \rho_{++}(\vec{r}_1, \vec{r}_2, t) - \rho_{+-}(\vec{r}_1, \vec{r}_2, t) \right\} \tau_{\rm ind}(\vec{r}_1 - \vec{r}_2) {\rm d}\vec{r}_2 \right] = 0$$

$$\frac{\partial \rho_-(\vec{r}_1, t)}{\partial t} - M_0 \vec{b} \frac{d}{d\vec{r}_1} \left[\rho_-(\vec{r}_1, t)\tau_{\rm ext} - \int \left\{ \rho_{--}(\vec{r}_1, \vec{r}_2, t) - \rho_{-+}(\vec{r}_1, \vec{r}_2, t) \right\} \tau_{\rm ind}(\vec{r}_1 - \vec{r}_2) {\rm d}\vec{r}_2 \right] = 0$$

Plastic shear

Dislocation density tensor

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with $\gamma = \beta^p_{12}$

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Geometrically Necessary Dislocation (GND) density

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Geometrically Necessary Dislocation (GND) density

Deformation rate

$$\frac{\partial \kappa}{\partial t} = -\frac{\vec{b}}{b^2}\frac{d\dot{\gamma}}{d\vec{r}}.$$

Self-consistent field

Simplest assumption

$$\rho_{ss'}(\vec{r}_1, \vec{r}_2, t) = \rho_s(\vec{r}_1)\rho_{s'}(\vec{r}_2), \; s, s' \in \{+, -\}.$$

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$$\frac{\partial \rho(\vec{r}, t)}{\partial t} + M_0 \vec{b} \frac{\partial}{\partial \vec{r}} [\kappa(\vec{r}, t) \{\tau_{sc}(\vec{r}, t) + \tau_{ext}\}] = 0$$

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where

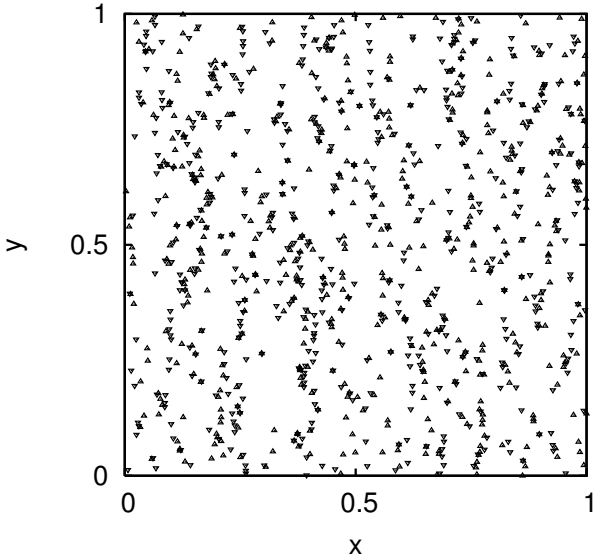
$$\tau_{sc}(\vec{r}) = \int \kappa(\vec{r}_1, t) \tau_{ind}(\vec{r} - \vec{r}_1) d\vec{r}_1$$

τ_{sc} is not "new"

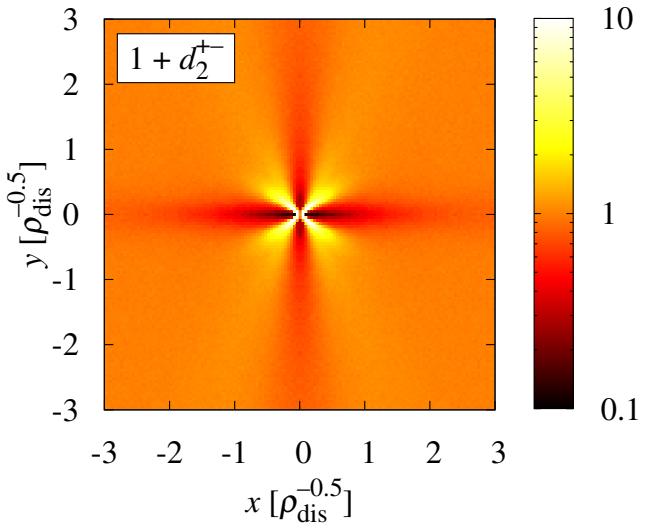
$$\Delta^2 \chi = \frac{2b\mu}{(1-\nu)} \frac{\partial}{\partial y} \kappa(\vec{r}), \qquad \tau_{sc} = \frac{\partial^2}{\partial x \partial y} \chi$$

Local density approximation

$$\rho_{ss'}(\vec{r}_1, \vec{r}_2, t) \quad = \quad \rho_s(\vec{r}_1)\rho_{s'}(\vec{r}_2)(1 + d_{ss'}(\vec{r}_1, \vec{r}_2)) \quad s, s' \in \{+, -\}$$



Relaxed dislocation configuration



Correlation function

Local density approximation

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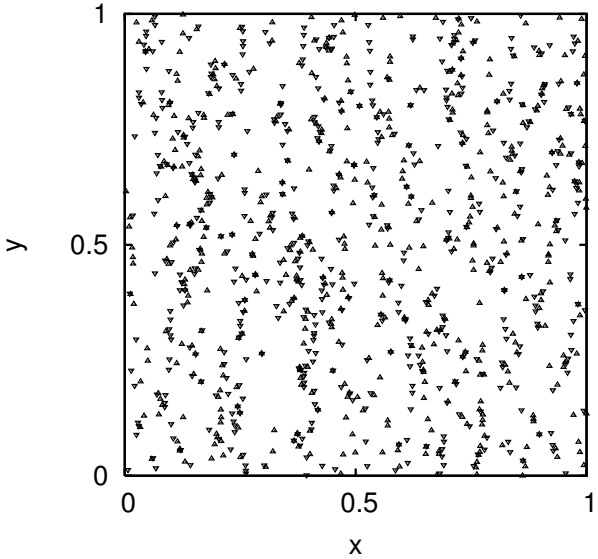

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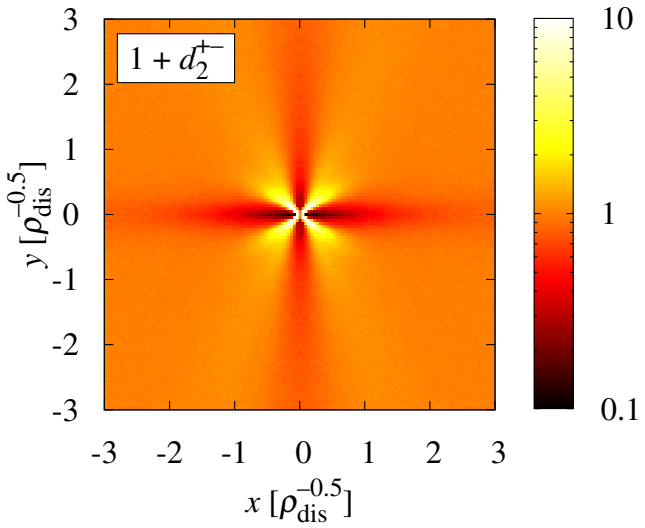

Correlation function

$$d_{ss'}(\vec{r}_1, \vec{r}_2) \quad = \quad d_{ss'}(\vec{r}_1 - \vec{r}_2, \rho(\vec{r}_1))$$

Local density approximation

$$\begin{aligned}
\rho_{ss'}(\vec{r}_1, \vec{r}_2, t) &= \rho_s(\vec{r}_1)\rho_{s'}(\vec{r}_2)(1 + d_{ss'}(\vec{r}_1, \vec{r}_2)) \quad s, s' \in \{+, -\} \\
d_{ss'}(\vec{r}_1, \vec{r}_2) &= d_{ss'}((\vec{r}_1 - \vec{r}_2)\sqrt{\rho(\vec{r}_1)})
\end{aligned}$$

Local density approximation

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Stress "like" terms with $\rho=\rho_+ + \rho_-$ and $\kappa = \rho_+ - \rho_-$

$$\partial_t\rho_+(\vec{r},t) + M_0 b\partial_x\left\{\rho_+\left[\tau_{\rm ext}+\tau_{\rm sc}-\tau_f+\tau_b+\tau_a+\frac{\kappa}{\rho}\tau_f\right]\right\}=0$$

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Correlation functions

$$\begin{aligned}d_s &= \tfrac{1}{2}(d_{++}+d_{--}+d_{+-}+d_{-+}),\\ d_p &= \tfrac{1}{2}(d_{++}+d_{--}-d_{+-}-d_{-+}),\\ d_a &= \tfrac{1}{2}(d_{++}-d_{--}-d_{+-}+d_{-+}),\\ d_{a'} &= \tfrac{1}{2}(d_{++}-d_{--}+d_{+-}-d_{-+}).\end{aligned}$$

Local density approximation

$$\begin{aligned}\rho_{ss'}(\vec{r}_1,\vec{r}_2,t) &= \rho_s(\vec{r}_1)\rho_{s'}(\vec{r}_2)(1+d_{ss'}(\vec{r}_1,\vec{r}_2)) \quad s,s'\in\{+,-\}\\ d_{ss'}(\vec{r}_1,\vec{r}_2) &= d_{ss'}((\vec{r}_1-\vec{r}_2)\sqrt{\rho(\vec{r}_1)})\end{aligned}$$

Stress "like" terms with $\rho=\rho_+ + \rho_-$ and $\kappa = \rho_+ - \rho_-$

$$\partial_t\rho_+(\vec{r},t) + M_0 b\partial_x\left\{\rho_+\left[\tau_{\rm ext}+\tau_{\rm sc}-\tau_f+\tau_b+\tau_a+\frac{\kappa}{\rho}\tau_f\right]\right\}=0$$

$$\partial_t\rho_-(\vec{r},t) - M_0 b\partial_x\left\{\rho_-\left[\tau_{\rm ext}+\tau_{\rm sc}-\tau_f+\tau_b-\tau_a-\frac{\kappa}{\rho}\tau_f\right]\right\}=0$$

$$\begin{aligned}\tau_{\rm sc}(\vec{r}) &= \int \tau_{\rm ind}(\vec{r}-\vec{r}')\kappa(\vec{r}')\mathrm{d}^2r', \quad \frac{1-\nu}{2\mu}\triangle^2\chi = b\partial_y\kappa\\ \tau_f(\vec{r}) &= -\int \rho(\vec{r}')d_a(\vec{r}-\vec{r}')\tau_{\rm ind}(\vec{r}-\vec{r}')\mathrm{d}^2r'\\ \tau_b(\vec{r}) &= \int \kappa(\vec{r}')d_s(\vec{r}-\vec{r}')\tau_{\rm ind}(\vec{r}-\vec{r}')\mathrm{d}^2r'\\ \tau_a(\vec{r}) &= \int \rho(\vec{r}')d_p(\vec{r}-\vec{r}')\tau_{\rm ind}(\vec{r}-\vec{r}')\mathrm{d}^2r'\end{aligned}$$

Local density approximation

$$\begin{aligned}
\rho_{ss'}(\vec{r}_1,\vec{r}_2,t) &= \rho_s(\vec{r}_1)\rho_{s'}(\vec{r}_2)(1+d_{ss'}(\vec{r}_1,\vec{r}_2)) \quad s,s' \in \{+,-\} \\
d_{ss'}(\vec{r}_1,\vec{r}_2) &= d_{ss'}((\vec{r}_1-\vec{r}_2)\sqrt{\rho(\vec{r}_1)})
\end{aligned}$$

Stress "like" terms with $\rho = \rho_+ + \rho_-$ and $\kappa = \rho_+ - \rho_-$

$$\partial_t \rho_+(\vec{r},t) + M_0 b \partial_x \left\{ \rho_+ \left[\tau_{\rm ext} + \tau_{\rm sc} - \tau_f + \tau_b + \tau_a + \frac{\kappa}{\rho}\tau_f \right] \right\} = 0$$

$$\partial_t \rho_-(\vec{r},t) - M_0 b \partial_x \left\{ \rho_- \left[\tau_{\rm ext} + \tau_{\rm sc} - \tau_f + \tau_b - \tau_a - \frac{\kappa}{\rho}\tau_f \right] \right\} = 0$$

$$\begin{aligned}
\tau_{\rm sc}(\vec{r}) &= \int \tau_{\rm ind}(\vec{r}-\vec{r}')\kappa(\vec{r}')\mathrm{d}^2 r', \quad \frac{1-\nu}{2\mu}\triangle^2 \chi = b\partial_y \kappa \\
\tau_f(\vec{r}) &= \text{Friction stress .... } \mu b \alpha \sqrt{\rho(\vec{r})}, \\
\tau_b(\vec{r}) &= -Gb\frac{D}{\rho}\partial_x \kappa(\vec{r}), \\
\tau_a(\vec{r}) &= -Gb\frac{A}{\rho}\partial_x \rho(\vec{r}),
\end{aligned}$$

Phase field theory

Balance equations for ρ_+ and ρ_-

$$\partial_t \rho_\pm + \partial_x[\rho_\pm v\pm] = \pm f(\rho_+, \rho_-)$$

Phase field theory

Balance equations for ρ_+ and ρ_- with no sources

$$\partial_t \rho_\pm + \partial_x [\rho_\pm v \pm] = 0$$

Phase field theory

Balance equations for ρ_+ and ρ_- with no sources

$$\partial_t \rho_\pm + \partial_x [\rho_\pm v\pm] = 0$$

Dissipative motion ($v_\pm \propto \mp\tau$)

$$\tau = \frac{\delta E}{\delta \gamma}, \quad E\left[\kappa = \frac{1}{b}\partial_x \gamma\right] \quad \Rightarrow \quad \tau = -\partial_x \frac{\delta E}{\delta \kappa}, \quad \Rightarrow \quad v_\pm = -M_0 \partial_x \frac{\delta E}{\delta \kappa}$$

Phase field theory

Balance equations for ρ_+ and ρ_- with no sources

$$\partial_t \rho_\pm + \partial_x [\rho_\pm v\pm] = 0$$

Dissipative motion ($v_\pm \propto \mp\tau$)

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Phase field functional $P[\rho_+, \rho_-, \chi]$

$$\begin{aligned} v_+ &= -M_0 \left\{ \partial_x \left[\frac{1+\beta}{2} \frac{\delta P}{\delta \rho_+} - \frac{1-\beta}{2} \frac{\delta P}{\delta \rho_-} \right] \right\} \\ v_- &= -M_0 \left\{ \partial_x \left[\frac{1+\beta}{2} \frac{\delta P}{\delta \rho_-} - \frac{1-\beta}{2} \frac{\delta P}{\delta \rho_+} \right] \right\} \end{aligned}$$

Phase field theory

Balance equations for ρ_+ and ρ_- with no sources

$$\partial_t \rho_\pm + \partial_x [\rho_\pm v\pm] = 0$$

Dissipative motion ($v_\pm \propto \mp\tau$)

$$\tau = \frac{\delta E}{\delta \gamma}, \quad E\left[\kappa = \frac{1}{b}\partial_x \gamma\right] \quad \Rightarrow \quad \tau = -\partial_x \frac{\delta E}{\delta \kappa}, \quad \Rightarrow \quad v_\pm = -M_0 \partial_x \frac{\delta E}{\delta \kappa}$$

Phase field functional $P[\rho_+, \rho_-, \chi]$

$$\begin{aligned} \partial_t \rho_+ - \partial_x \left\{ \rho_+ M_0 \left[\partial_x \frac{\delta P}{\delta \kappa} + \beta \partial_x \frac{\delta P}{\delta \rho} \right] \right\} &= 0 \\ \partial_t \rho_- + \partial_x \left\{ \rho_- M_0 \left[\partial_x \frac{\delta P}{\delta \kappa} - \beta \partial_x \frac{\delta P}{\delta \rho} \right] \right\} &= 0 \end{aligned}$$

Phase field functional

Coarse grained energy functional

$$P_{\mathrm{sc}}[\chi, \rho_+, \rho_-] = \int \left[-\frac{1-\nu}{4\mu} (\triangle \chi)^2 + b\chi \partial_y \kappa \right] \mathrm{d}^2 r$$

Phase field functional

Coarse grained energy functional

$$P_{\rm sc}[\chi,\rho_+,\rho_-] = \int \left[-\frac{1-\nu}{4\mu}(\triangle\chi)^2 + b\chi\partial_y\kappa \right] \mathrm{d}^2 r$$

Minimum condition

$$\begin{aligned} \frac{\delta P_{\rm sc}}{\delta\chi} &= 0 \\ \frac{1-\nu}{2\mu}\triangle^2\chi &= b\partial_y\kappa, \quad \tau_{\rm sc} = \partial_x\partial_y\chi \end{aligned}$$

Phase field functional

Coarse grained energy functional

$$P_{\rm sc}[\chi,\rho_+,\rho_-] = \int \left[-\frac{1-\nu}{4\mu}(\triangle\chi)^2 + b\chi\partial_y\kappa \right] \mathrm{d}^2 r$$

Minimum condition

$$\begin{aligned} \frac{\delta P_{\rm sc}}{\delta\chi} &= 0 \\ \frac{1-\nu}{2\mu}\triangle^2\chi &= b\partial_y\kappa, \quad \tau_{\rm sc} = \partial_x\partial_y\chi \end{aligned}$$

Evolution equation

$$\begin{aligned} \frac{\partial\rho(\vec{r},t)}{\partial t} + M_0\vec{b}\frac{\partial}{\partial\vec{r}}[\kappa(\vec{r},t)\{\tau_{sc}(\vec{r},t)+\tau_{ext}\}] &= 0 \\ \frac{\partial\kappa(\vec{r},t)}{\partial t} + M_0\vec{b}\frac{\partial}{\partial\vec{r}}[\rho(\vec{r},t)\{\tau_{sc}(\vec{r},t)+\tau_{ext}\}] &= 0 \end{aligned}$$

Correlation part

Correlation contribution

$P[\rho_+, \rho_-, \chi] = P_{\mathrm{sc}}[\rho_+, \rho_-, \chi] + P_{\mathrm{corr}}[\rho_+, \rho_-]$

$$P^{\pm}_{\mathrm{corr}}[\rho_\pm] = \int \left[GAb\rho \ln\left(\frac{\rho}{\rho_0}\right) + \frac{GDb}{2}\frac{\kappa^2}{\rho} \right] \mathrm{d}^2 r$$

Correlation part

Correlation contribution

$P[\rho_+, \rho_-, \chi] = P_{\rm sc}[\rho_+, \rho_-, \chi] + P_{\rm corr}[\rho_+, \rho_-]$

$$P^{\pm}_{\rm corr}[\rho_\pm] = \int \left[GAb\rho \ln\left(\frac{\rho}{\rho_0}\right) + \frac{GDb}{2}\frac{\kappa^2}{\rho} \right] {\rm d}^2 r$$

Evolution equations

$$\partial_t \rho = -Mb\partial_x \left\{ \kappa\tau_{\rm sc} - GbD\frac{\kappa}{\rho}\partial_x\kappa - GbA\partial_x\rho \right\}$$

$$\partial_t \kappa = -Mb\partial_x \left\{ \rho\tau_{\rm sc} - GbD\partial_x\kappa - GbA\frac{\kappa}{\rho}\partial_x\rho \right\}$$

Flow stress

Adding flow stress

$$\partial_t \rho = -Mb\partial_x \left\{ \kappa \tau_{\rm sc} - GbD\frac{\kappa}{\rho}\partial_x \kappa - GbA\partial_x \rho \right\}$$

$$\partial_t \kappa = -Mb\partial_x \quad \left\{ \rho \tau_{\rm sc} - \rho \left(1 - \frac{\kappa^2}{\rho^2} \right) \tau_{\rm f} - GbD\partial_x \kappa - GbA\frac{\kappa}{\rho}\partial_x \rho \right\}$$

$$\tau_f(\tau_{sc}) = \begin{cases} \tau_{sc} & \text{if} \quad \tau_{sc} < \alpha\mu b\sqrt{\rho} \\ \alpha\mu b\sqrt{\rho} & \text{if} \quad \tau_{sc} > \alpha\mu b\sqrt{\rho} \end{cases}$$

Flow stress

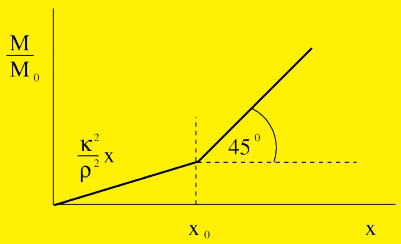
Adding flow stress

$$\partial_t \rho = \partial_x \left\{ \kappa M_0 \partial_x \frac{\delta P}{\delta \kappa} + \rho M_0 \partial_x \frac{\delta P}{\delta \rho} \right\}$$

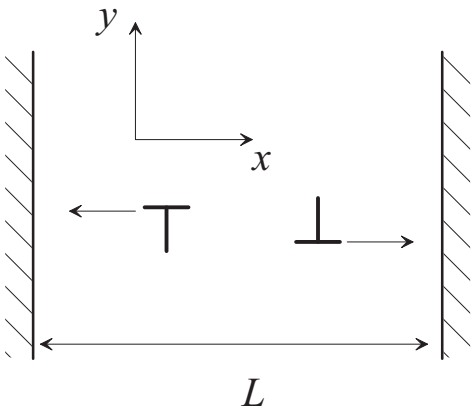
$$\partial_t \kappa = \partial_x \left\{ \rho M \left(\partial_x \frac{\delta P}{\delta \kappa} \right) + \kappa M_0 \partial_x \frac{\delta P}{\delta \rho} \right\}$$

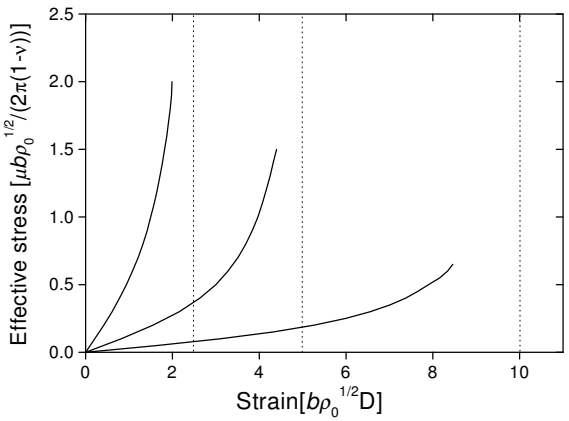
$$M(x) = M_0 \begin{cases} \frac{\kappa^2}{\rho^2} x & \text{if} \quad |x| < x_0 \\ \mathrm{sgn}(x) \left[|x| - x_0 \left(1 - \frac{\kappa^2}{\rho^2} \right) \right] & \text{if} \quad |x| \geqq x_0 \end{cases}$$

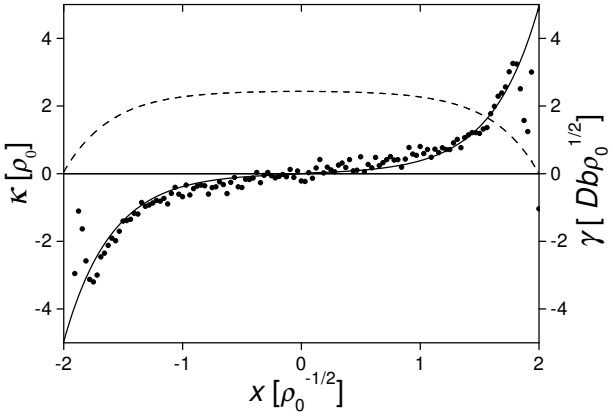
with $x_0 = \alpha \mu b^2 \sqrt{\rho}$



Finite size effect







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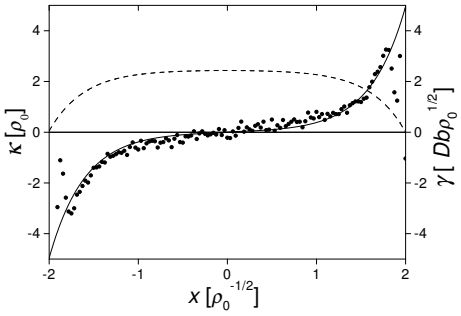
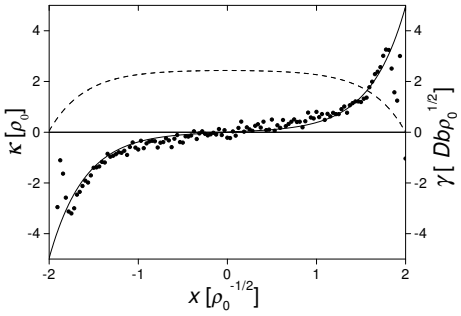


Finite size effect

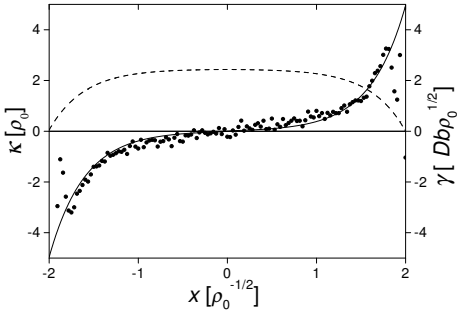


if $\rho = 1/l^2$

$$\tau = GbD \frac{1}{\rho(\vec{r})} \frac{\partial \kappa(\vec{r})}{\partial x}$$

$$\kappa(\vec{r}) = \tau \frac{1}{l^2} \frac{1}{GbD} x$$

Finite size effect



if $\rho = 1/l^2$

$$\tau = GbD \frac{1}{\rho(\vec{r})} \frac{\partial \kappa(\vec{r})}{\partial x}$$

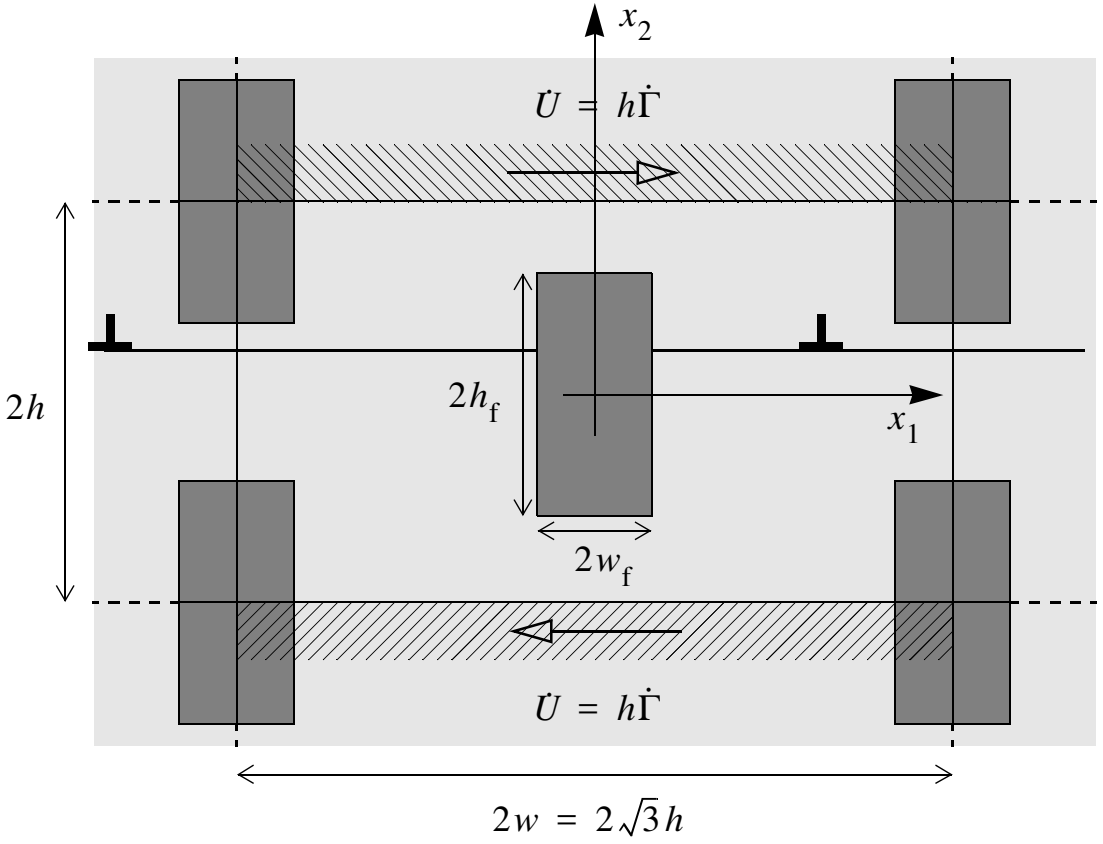
$$\kappa(\vec{r}) = \tau \frac{1}{l^2} \frac{1}{GbD} x$$

at the surface $-L/2$ $\rho = -\kappa$

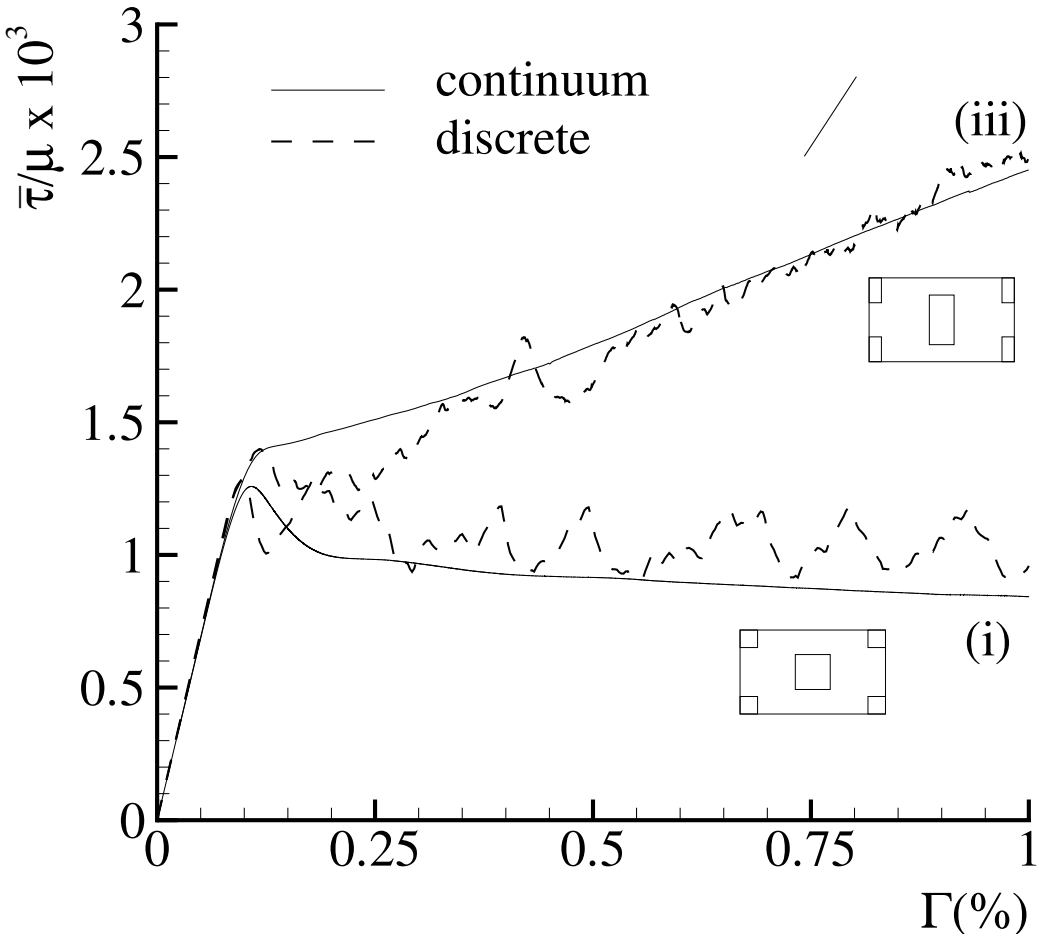
$$\tau = -GbD \frac{1}{\kappa(\vec{r})} \frac{\partial \kappa(\vec{r})}{\partial x}$$

$$\kappa(\vec{r}) = \kappa_0 \exp\left\{-\tau \frac{1}{GbD} x\right\}$$

NG composite



NG composite



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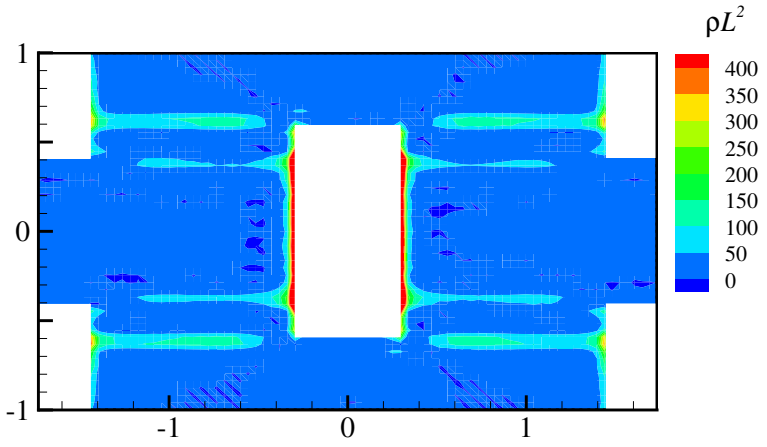

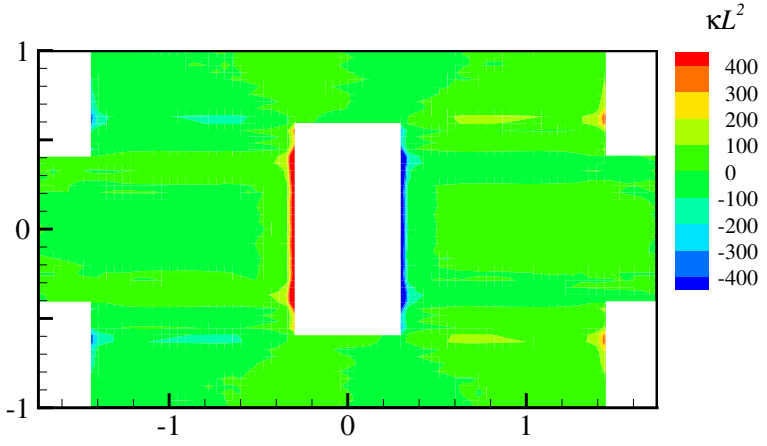

Debye screening

Single dislocation

$$\frac{1-\nu}{2\mu} \triangle^2 \chi = b\partial_2 \delta$$

Debye screening

Single dislocation+"bath"

$$\frac{1-\nu}{2\mu}\triangle^2\chi = b\partial_2\delta + b\partial_2\kappa$$

Debye screening

Single dislocation+"bath"

$$\frac{1-\nu}{2\mu}\triangle^2\chi = b\partial_2\delta + b\partial_2\kappa$$

Single dislocation+induced GND

$$\frac{1-\nu}{2\mu}\triangle^2\chi = b\partial_2\left\{\delta + \frac{\rho_0}{GbD}\partial_2\chi\right\}$$

Debye screening

Single dislocation+"bath"

$$\frac{1-\nu}{2\mu}\triangle^2\chi = b\partial_2\delta + b\partial_2\kappa$$

Single dislocation+induced GND

$$\frac{1-\nu}{2\mu}\triangle^2\chi = b\partial_2\left\{\delta + \frac{\rho_0}{GbD}\partial_2\chi\right\}$$

Solution

$$\chi \propto \sinh(k_0 y)K_0(k_0 r)$$

Debye screening

Single dislocation+"bath"

$$\frac{1-\nu}{2\mu}\triangle^2\chi = b\partial_2\delta + b\partial_2\kappa$$

Single dislocation+induced GND

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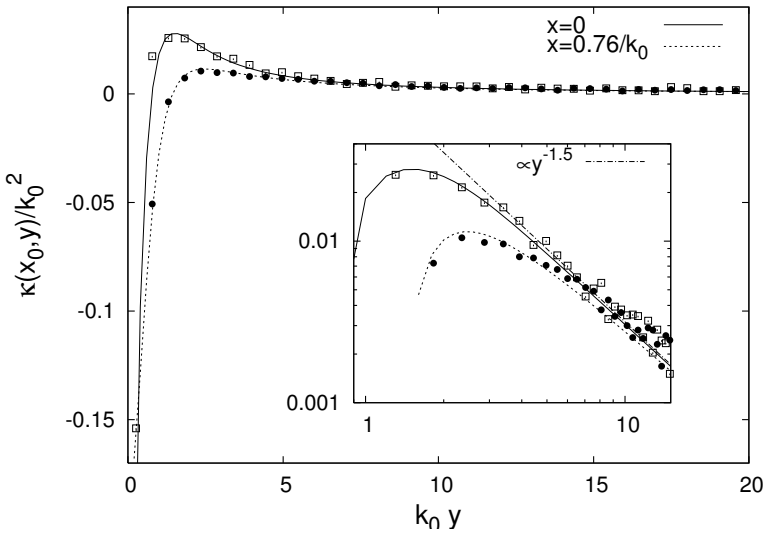
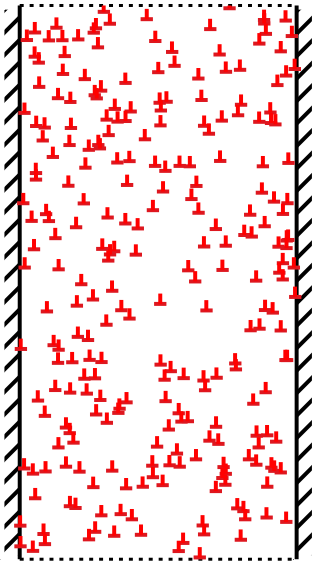
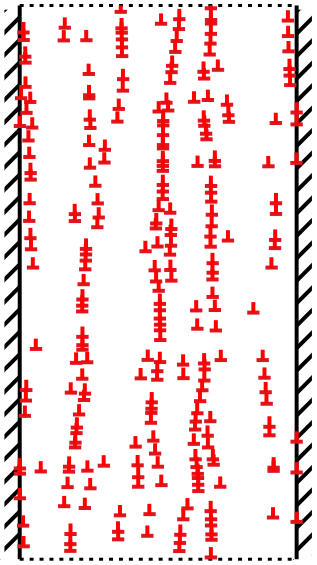


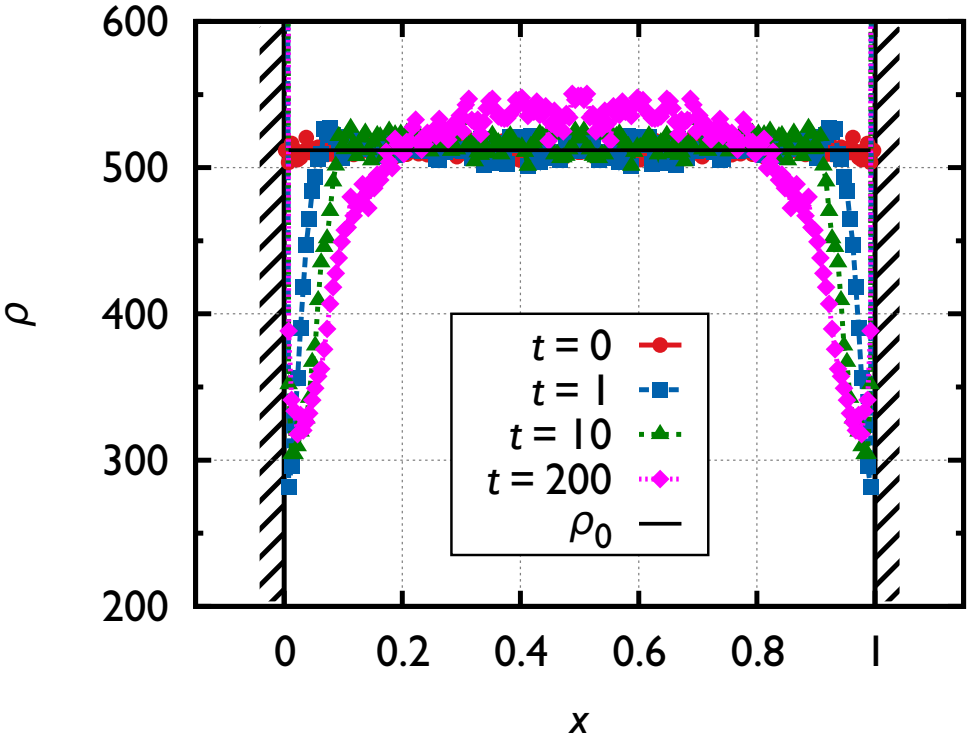
Only GND



Only GND

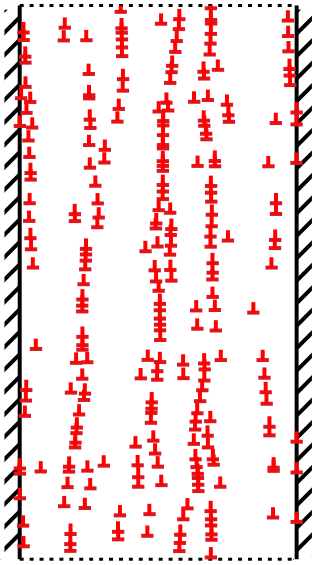


Dislocation configuration

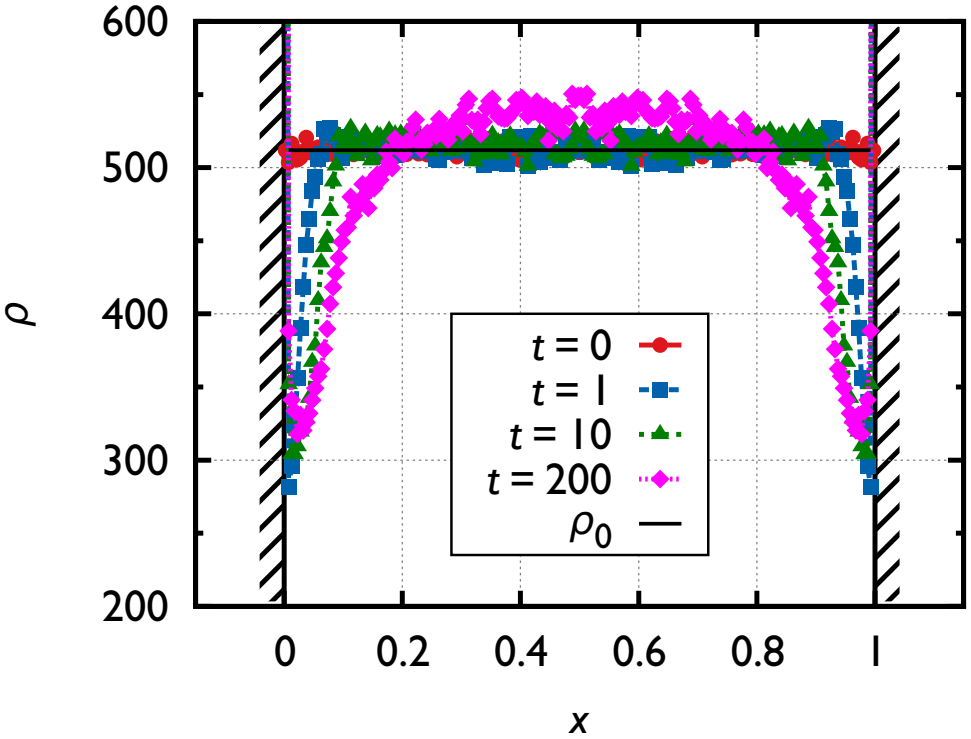


Density evolution

Only GND



Dislocation configuration



Density evolution

$$j_+ = -M_0 A b G \partial_x \rho_+ \equiv 0$$

Beyond Gradient term

Plastic potential

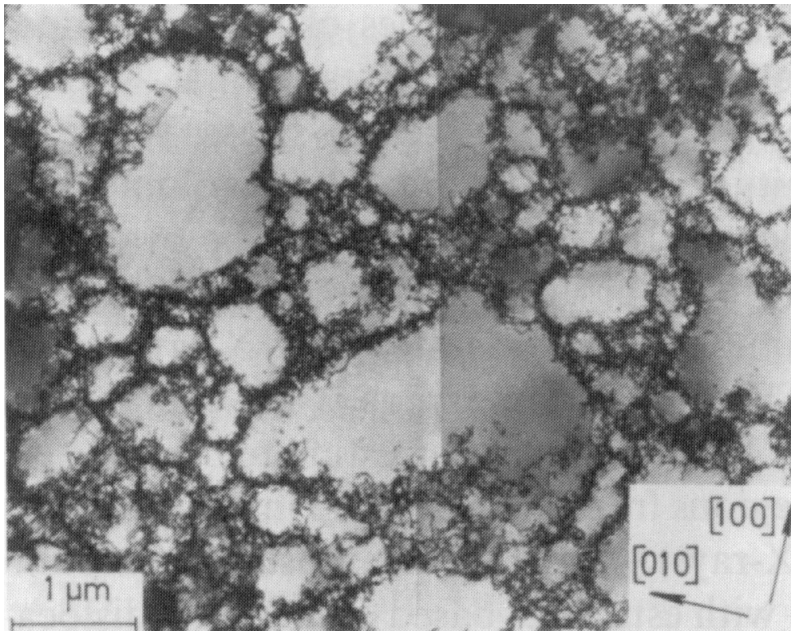
$$P = \int_G \left\{ -\frac{(1-\nu}{4\mu}(\triangle\chi)^2 + b\chi\partial_y\rho_+ + GbA\rho_+ \ln(\rho_+/\rho_0) + \rho_+ U\left[(\partial_x\rho_+)^2/\rho_+^3\right] \right\} \mathrm{d}^2 r$$

Beyond Gradient term

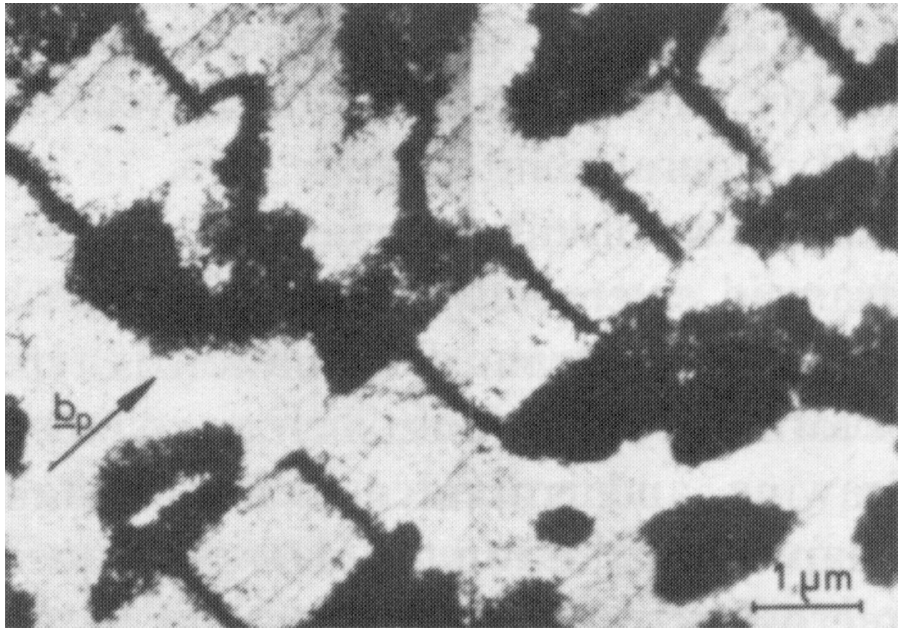
Surface term $\Rightarrow$ Boundary condition

$$P_{surface} = \int_{\partial G} \lambda \sqrt{\rho_+} (\vec{b}\vec{n}) \mathrm{ds}$$

Dislocation patterning



Cell structure



PSB structure

Stability analysis

Trivial solution

$$\begin{aligned}\rho(\vec{r},t) &= \rho + \delta\rho(\vec{r},t)\\ \kappa(\vec{r},t) &= \delta\kappa(\vec{r},t)\\ \chi(\vec{r},t) &= \tau_0 xy + \delta\chi(\vec{r},t)\end{aligned}$$

Periodic perturbation

$$\begin{pmatrix}\delta\rho\\ \delta\kappa\\ \delta\chi\end{pmatrix} = \begin{pmatrix}\delta\rho_0\\ \delta\kappa_0\\ \delta\chi_0\end{pmatrix} \exp\left(\frac{\lambda}{t_0}t + i\sqrt{\rho_0}\vec{k}\vec{r}\right)$$

Stability analysis

Trivial solution

$$\begin{aligned}\rho(\vec{r},t) &= \rho + \delta\rho(\vec{r},t)\\ \kappa(\vec{r},t) &= \delta\kappa(\vec{r},t)\\ \chi(\vec{r},t) &= \tau_0 xy + \delta\chi(\vec{r},t)\end{aligned}$$

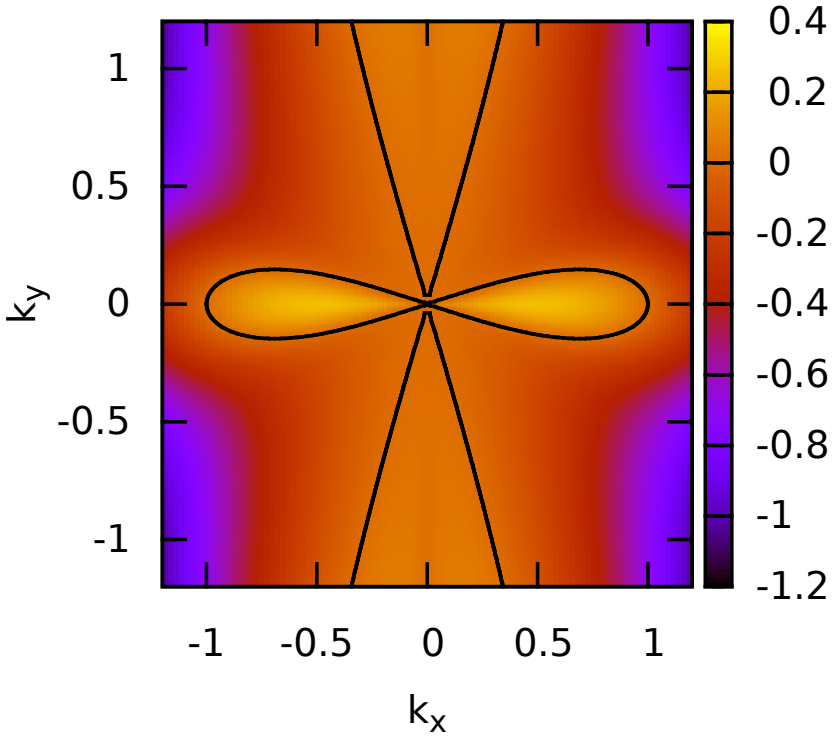
Periodic perturbation

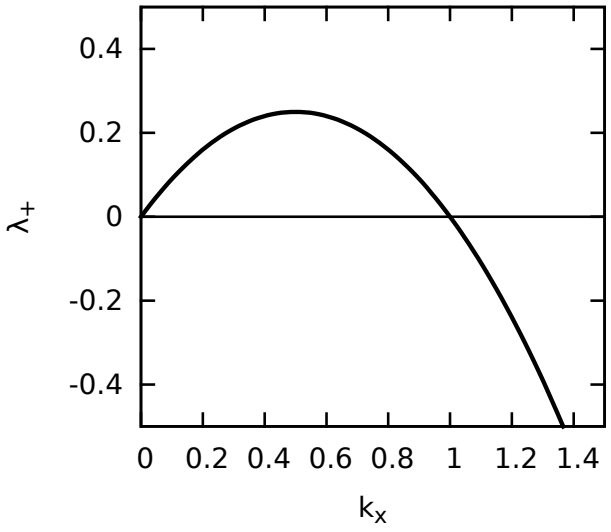
$$\begin{pmatrix}\delta\rho\\ \delta\kappa\\ \delta\chi\end{pmatrix} = \begin{pmatrix}\delta\rho_0\\ \delta\kappa_0\\ \delta\chi_0\end{pmatrix} \exp\left(\frac{\lambda}{t_0}t + i\sqrt{\rho_0}\vec{k}\vec{r}\right)$$

Stability condition

$$(\dot{\gamma}' + 2\alpha')(\dot{\gamma}' - \alpha') + A\frac{k_x^2 k_y^2}{k^4} + ADk_x^2 < 0.$$

Length scale selection





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Stuart Pawley
Gilles Canova
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3D

Fields: $\rho(\vec{r},\varphi)$ and $k(\vec{r},\varphi)$

$$\hat{\nabla} = \rho(\vec{r},\varphi) + \partial_\phi$$

$$L(\vec{r},\varphi) = (\cos(\phi), \sin(\phi), 0, k(\vec{r},\varphi))$$

Velocities

$$\vec{v} = (v_1, v_2, 0) = v(\vec{r},\varphi)(\sin(\phi), -\cos(\phi), 0)$$

$$V(\vec{r},\varphi) = (v_1, v_2, 0, -\nabla_L v)$$

Evolution eqs. $q = \rho k$

$$\partial_t \rho = -\hat{\nabla}(\rho V) + qv$$

$$\partial_t q = -\hat{\nabla}(qV) - \rho \hat{\nabla}_L \hat{\nabla}_L v$$